Problem Set 7 due November 4, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

(1) Find a basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $\mathbb{R}^2$ such that $f(\boldsymbol{v}_1) = \boldsymbol{e}_2$ and $f(\boldsymbol{v}_2) = \boldsymbol{e}_3$, where $\boldsymbol{e}_i$ is the i–th coordinate unit vector. Compute the matrix B which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2) = (b_{11}x_1 + b_{12}x_2)\boldsymbol{e}_1 + (b_{21}x_1 + b_{22}x_2)\boldsymbol{e}_2 + (b_{31}x_1 + b_{32}x_2)\boldsymbol{e}_3$$

and say explicitly how B relates to A (hint: B should be equal to A times a matrix). *(10 points)*

Solution: Finding $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ comes down to solving a system of equations, and it's easy to see that the solution is $\boldsymbol{v}_1 = -\frac{1}{5}(\boldsymbol{e}_1 - 3\boldsymbol{e}_2)$ and $\boldsymbol{v}_2 = -\frac{1}{5}(\boldsymbol{e}_2 - 2\boldsymbol{e}_1)$. Then,

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix};$$

letting $X = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}$, we have that $B = AX$.

Grading Rubric: 5 points for the basis of $\mathbb{R}^2$: 5/5 if completely correct, 2/5 if somewhat correct (perhaps off by a scalar), 0/5 if mostly incorrect. 2 points for B: 2/2 if correct, 0/2 if incorrect. 3 points for relating B and A: 3/3 if correct, 0/3 if incorrect.

(2) Find a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ of $\mathbb{R}^3$ such that $f(\boldsymbol{e}_1) = \boldsymbol{w}_1$ and $f(\boldsymbol{e}_2) = \boldsymbol{w}_2$. Compute the matrix C which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) = (c_{11}x_1 + c_{12}x_2)\boldsymbol{w}_1 + (c_{21}x_1 + c_{22}x_2)\boldsymbol{w}_2 + (c_{31}x_1 + c_{32}x_2)\boldsymbol{w}_3$$

and say explicitly how C relates to A (hint: C should be equal to A times a matrix). *(10 points)*

Solution: Let $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\boldsymbol{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix};$$

letting $Y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & \frac{3}{5} & -\frac{1}{5} \\ 1 & 0 & 0 \end{bmatrix}$, we have that $C = YA$.

Grading Rubric: 5 points for a basis of $\mathbb{R}^3$: 5/5 if completely correct, 2/5 if somewhat correct (perhaps off by a scalar or if $\boldsymbol{w}_3$ does not actually complete $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ to a basis), 0/5 if mostly incorrect. 2 points for C: 2/2 if correct, 0/2 if incorrect. 3 points for relating C and A: 3/3 if correct, 0/3 if incorrect.

Problem 2: Let A_n be the $n \times n$ matrix with 2's on the diagonal, and -1's directly above and directly below. For example:

$$A_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(1) Compute $\det A_2$, $\det A_3$, $\det A_4$ by <u>row operations</u> (i.e. putting the matrix in question in row echelon form, multiplying the pivots then multiplying by $(-1)^{\#\text{ of row exchanges performed}}$). *(10 points)*

Solution: We have

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & \frac{3}{2} \end{bmatrix},$$

so $\det A_2 = 2 \cdot \frac{3}{2} = 3$. Similarly,

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

so $\det A_3 = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} = 4$, and

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

so $\det A_4 = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = 5$.

Grading Rubric: 2 pts. for $\det A_2$: 2/2 for a completely correct computation, 0/2 if incorrect. 4 pts. each for $\det A_3$ and $\det A_4$: 4/4 for a completely correct computation, 2/4 if slightly wrong (e.g. sign error, slightly wrong row echelon form), 0/4 for mostly incorrect attempt.

(2) Use <u>cofactor expansion</u> along the first row to obtain a recursive formula for $\det A_n$ in terms of $\det A_{n-1}$ and $\det A_{n-2}$, for all natural numbers n. *(10 points)*

Solution: Using cofactor expansion (twice), we see that

$$\det A_n = 2 \det A_{n-1} + \det \begin{bmatrix} -1 & \textbf{-1} \\ \mathbf{0} & A_{n-2} \end{bmatrix} = 2 \det A_{n-1} - \det A_{n-2}$$

where **-1** $= \begin{bmatrix} -1 & 0 & \cdots & 0 \end{bmatrix}$ (a $1 \times (n-2)$ matrix) and $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (an $(n-2) \times 1$ matrix).

Grading Rubric: 10/10 if completely correct, 5/10 if there is a sign error in the second term of the expansion, 0/10 for mostly incorrect attempt.

(3) Guess what $\det A_n$ is in general, and use the recursive formula in part (2) to prove your guess. *(5 points)*

Solution: We will show that $\det A_n = n+1$, by induction. For $n = 1$, we have $A_1 = 2$ so $\det A_1 = 2$; for $n = 2$, we have from (1) above that $\det A_2 = 3$, as desired. Now, suppose $\det A_k = k+1$ for all $k \leq n$ for some $n \geq 3$. Then from (2) we have

$$\det A_{n+1} = 2 \det A_n - \det A_{n-1} = 2(n+1) - n = n+2,$$

as desired.

Grading Rubric: 5/5 if completely correct, 3/5 for subtle error in induction proof, 0/5 for mostly incorrect attempt (including if the formula for $\det A_n$ was incorrect to begin with).

Problem 3: Let $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ be the 3×3 matrix defined in the previous problem.

(1) Use the cofactor formula for the inverse to compute A_3^{-1}. *(10 points)*

Solution: We first find the cofactor matrix, C_3:

$$C_3 = \begin{bmatrix} 2^2 - (-1)^2 & (-1)(-2-0) & (-1)^2 - 0 \\ (-1)(-2-0) & 2^2 - 0 & (-1)(-2-0) \\ (-1)^2 - 0 & (-1)(-2-0) & 2^2 - (-1)^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Then,

$$A_3^{-1} = \frac{1}{\det A_3} C_3^T = \frac{1}{4} C_3 = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Grading Rubric: 10/10 if completely correct, 5/10 if correct up to sign or if off by factor of $\det A_3$, 0/10 for mostly incorrect attempt.

(2) Use Cramer's rule to compute the solution to the equation:

$$A_3 \boldsymbol{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for any numbers b_1, b_2, b_3. *(5 points)*

Solution Let $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then, Cramer's rule and cofactor expansions of determinants give that

$$v_1 = \frac{1}{\det A_3} \det \begin{bmatrix} b_1 & -1 & 0 \\ b_2 & 2 & -1 \\ b_3 & -1 & 2 \end{bmatrix} = \frac{1}{4}(b_1(3) + (2b_2 + b_3)) = \frac{3b_1 + 2b_2 + b_3}{4},$$

$$v_2 = \frac{1}{\det A_3} \begin{bmatrix} 2 & b_1 & 0 \\ -1 & b_2 & -1 \\ 0 & b_3 & 2 \end{bmatrix} = \frac{1}{4}(2(2b_2 + b_3) - b_1(-2)) = \frac{b_1 + 2b_2 + b_3}{2},$$

and

$$v_3 = \frac{1}{\det A_3} \begin{bmatrix} 2 & -1 & b_1 \\ -1 & 2 & b_2 \\ 0 & -1 & b_3 \end{bmatrix} = \frac{1}{4}(2(2b_3 + b_2) + (-b_3 + b_1)) = \frac{b_1 + 2b_2 + 3b_3}{4},$$

which is consistent with (1).

Grading Rubric: 5/5 if completely correct, 2/5 if off by factor of $\det A_3$, 0/5 for mostly incorrect attempt.

Problem 4: Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be any two vectors in $\mathbb{R}^n$ which are <u>not</u> orthogonal.

(1) What is the rank of the matrix $A = \boldsymbol{v}\boldsymbol{w}^T$? *(5 points)*

Solution: The rank of A is 1. Indeed, for any $\boldsymbol{u} \in \mathbb{R}^n$, we have $A\boldsymbol{u} = \boldsymbol{v}\boldsymbol{w}^T\boldsymbol{u} = (\boldsymbol{w} \cdot \boldsymbol{u})\boldsymbol{v}$, so $C(A)$ is spanned by $\boldsymbol{v}$ (which is nonzero because the vectors are not orthogonal).

Grading Rubric: 5/5 if completely correct (ideally mentioning that $\boldsymbol{v}$ is nonzero), 0/5 if incorrect.

(2) Show that $\boldsymbol{v}$ is an eigenvector of the matrix A. What is the corresponding eigenvalue? *(5 points)*

Solution: We see that

$$A\boldsymbol{v} = \boldsymbol{v}\boldsymbol{w}^T\boldsymbol{v} = (\boldsymbol{w} \cdot \boldsymbol{v})\boldsymbol{v},$$

so $\boldsymbol{v}$ is an eigenvector with corresponding eigenvalue $\boldsymbol{w} \cdot \boldsymbol{v}$.

Grading Rubric: 5/5 if completely correct, 0/5 if incorrect.

(3) Find $n-1$ other eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}$, which together with $\boldsymbol{v}$ form a basis of $\mathbb{R}^n$ (your choice will depend on $\boldsymbol{w}$). What are the corresponding eigenvalues of these $n-1$ eigenvectors? *(10 points)*

Solution: Because all vectors $A\boldsymbol{u}$ are multiples of $\boldsymbol{v}$, the other $n-1$ eigenvectors must have eigenvalue 0 (i.e. they are in the null space of A). By the definition of A, the null space consists of vectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ orthogonal to $\boldsymbol{w}$, i.e. such that $w_1x_1 + w_2x_2 + \cdots + w_nx_n = 0$. Let j be the smallest integer such that $w_j \neq 0$. Then, we see that $x_1, \dots, x_{j-1}$ are free variables and then $x_j + \frac{w_{j+1}}{w_j} x_{j+1} + \cdots + \frac{w_n}{w_j} x_n = 0$. Thus letting $\boldsymbol{v}_k = \boldsymbol{e}_k$ for $1 \leq k < j$ and $\boldsymbol{v}_l = \boldsymbol{e}_{l+1} - \frac{w_{l+1}}{w_j} \boldsymbol{e}_j$ for $j \leq l \leq n-1$, we have that $\{\boldsymbol{v}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}\}$ is a basis for $\mathbb{R}^n$.

Grading Rubric: 3 points for recognizing that the eigenvalue is 0: 3/3 if correct, 0/3 if incorrect. 7 points for the $n-1$ vectors: 7/7 if the vectors work as a choice for a basis of the null space, 4/7 if choice of vectors is somewhat correct, 0/7 if mostly incorrect.

Problem 5: Consider the following block matrix (where A, B, C are 2×2 blocks):

$$X = \left[\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 2 & 0 & 3 \\ -3 & 5 & 9 & 0 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

(1) Compute the eigenvalues and eigenvectors of A and B. *(10 points)*

Solution: The characteristic polynomial of A is

$$p_A(\lambda) = \det(A - \lambda I) = -\lambda(5 - \lambda) + 6 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

so the eigenvalues of A are 2 and 3. Because

$$A - 2I = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix},$$

the eigenvectors with eigenvalue 2 are of the form $\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$, and because

$$A - 3I = \begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix},$$

the eigenvectors of eigenvalue 3 are of the form $\alpha \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$.

The characteristic polynomial of B is

$$p_B(\lambda) = \det(B - \lambda I) = (2 - \lambda)(-2 - \lambda) + 3 = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$$

so the eigenvalues of B are ± 1. Because

$$B - I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix},$$

the eigenvectors with eigenvalue 1 are of the form $\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $\beta \in \mathbb{R}$, and because

$$B + I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix},$$

the eigenvectors with eigenvalue -1 are of the form $\beta \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for some $\beta \in \mathbb{R}$.

Grading Rubric: 5 points for each matrix: 5/5 if eigenvalues and eigenvectors are correct, 2/5 if there is a mistake in computing the eigenvectors, 0/5 if eigenvalues and eigenvectors are both incorrect.

(2) Compute the characteristic polynomial and the eigenvalues of X (Hint: remember part (1)). What is the relationship between $\det X$, $\det A$, and $\det B$? *(10 points)*

Solution: We have that $\det X = (\det A)(\det B)$, so similarly $\det(X-\lambda I) = \det(A-\lambda I)\det(B-\lambda I)$. Thus the characteristic polynomial of X is

$$p_X(\lambda) = p_A(\lambda)p_B(\lambda) = (\lambda-2)(\lambda-3)(\lambda+1)(\lambda-1),$$

so the eigenvalues are $2, 3, -1$, and 1.

Grading Rubric: 5 points for identifying that $\det X = (\det A)(\det B)$: 5/5 if correct, 0/5 if incorrect. 5 points for $p_X(\lambda)$: 5/5 if correct, 0/5 if incorrect.